1. A curve C has parametric equations

$$x = 2t - 1, \qquad y = 4t - 7 + \frac{3}{t}, \qquad t \neq 0$$

Show that the Cartesian equation of the curve C can be written in the form

$$y = \frac{2x^2 + ax + b}{x + 1}, \qquad x \neq -1$$

where a and b are integers to be found.

(3)

$C: x = 2t - 1$, $y = 4t - 7 + \frac{3}{t} \Rightarrow y = 4\left(\frac{x+1}{2}\right) - 7 + \frac{3 \times 2}{x+1}$

$x + 1 = 2t$

$\Rightarrow t = \frac{x+1}{2}$

$\Rightarrow y = \frac{4(x+1)}{2} - 7 + \frac{6}{x+1}$ (1)

$\Rightarrow y = 2x + 2 - 7 + \frac{6}{x+1}$

$\Rightarrow y = (2x - 5) + \frac{6}{x+1}$

$\Rightarrow y = \frac{(2x-5)(x+1) + 6}{x+1}$ $\quad 2x - 5x = -3x$

$\Rightarrow y = \frac{2x^2 - 3x - 5 + 6}{x+1} = \frac{2x^2 - 3x + 1}{x+1}$ (1)

$y = \frac{2x^2 - 3x + 1}{x+1}$, $a = -3$ and $b = 1$

2. A curve C has parametric equations

$$x = 3 + 2\sin t, \qquad y = 4 + 2\cos 2t, \qquad 0 \leqslant t < 2\pi$$

(a) Show that all points on C satisfy $y = 6 - (x - 3)^2$ → Cartesian

(2)

(b) (i) Sketch the curve C.

→ Parabola y ∩

(ii) Explain briefly why C does not include all points of $y = 6 - (x - 3)^2$, $x \in \mathbb{R}$

(3)

The line with equation $x + y = k$, where k is a constant, intersects C at two distinct points.

(c) State the range of values of k, writing your answer in set notation.

(5)

a)

$\cos(2t) = 1 - 2\sin^2 t$

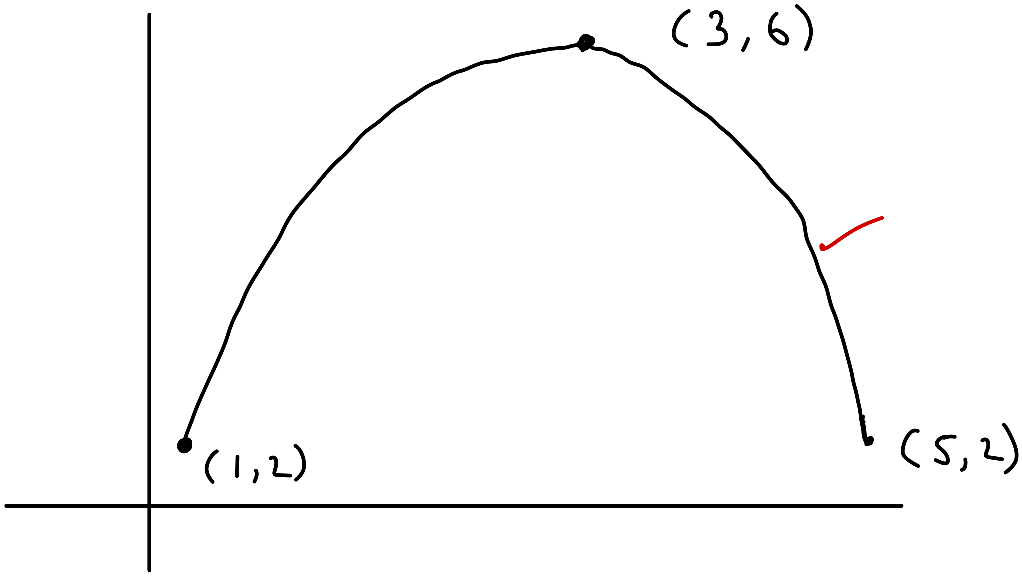
$y = 4 + 2(1 - 2\sin^2 t)$ ✓ $\qquad x = 3 + 2\sin t$

$= 4 + 2 \cdot 4\sin^2 t$ $\qquad 2\sin t = x - 3$

$y = 6 - (2\sin t)^2$

$\therefore y = 6 - (x - 3)^2$ as required. ✓

b)

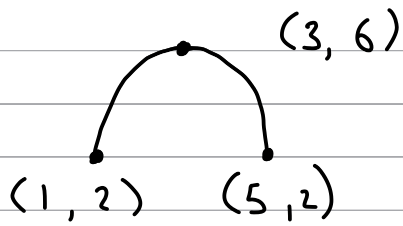


correct points = ✓

$y = 6 - (x-3)^2$ $\quad 0 \le t < 2\pi$

$x = 3 + 2\sin t$

$y = 4 + 2\cos(2t)$



max $y = 4 + 2\cos(0)$

$= 4 + 2 = 6$

min y: $\cos(2t) = -1$

$2t = \pi$

$t = \frac{\pi}{2}$

$\therefore$ min $y = 4 + 2(-1)$

$= 4 - 2 = 2$

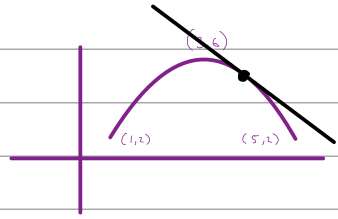
min x: $\sin(2t) = -1 \Rightarrow$ min $x = 3 - 2 = 1$

max : $\sin(2t) = 1 \Rightarrow$ max $x = 3 + 2 = 5$

ii) Since $0 \le t < 2\pi$, $x = 3 + 2\sin t$ can only take when $\sin t$ is $[-1, 1]$ $\therefore 1 \le x \le 5$ ✓

c)

$y = 6 - (x-3)^2$



The line takes equation $x+y=k$

first time 2 intersections occur : @ $(5,2)$

for $x+y=k$ @ $(5,2)$ $\Rightarrow k = 5+2 = 7$

Since $x+y=k$, $y = k-x$. $y = 6-(x-3)^2$

$$6-(x-3)^2 = k-x$$
$$6-(x^2-6x+9) = k-x$$
$$6-x^2+6x-9 = k-x$$
$$-x^2+7x-3-k = 0$$
$$x^2-7x+3+k = 0$$

final k value is when $b^2-4ac = 0$

$a = 1$
$b = -7$
$c = 3+k$

$$49 - 4(1)(3+k) = 0$$
$$49 - 12 - 4k = 0$$
$$37 - 4k = 0$$
$$4k = 37$$
$$k = \frac{37}{4}$$

range of 2 distinct points is $7 \leq k < \frac{37}{4}$

Range of values for $k = \left\{ k : 7 \leq k < \frac{37}{4} \right\}$

3.

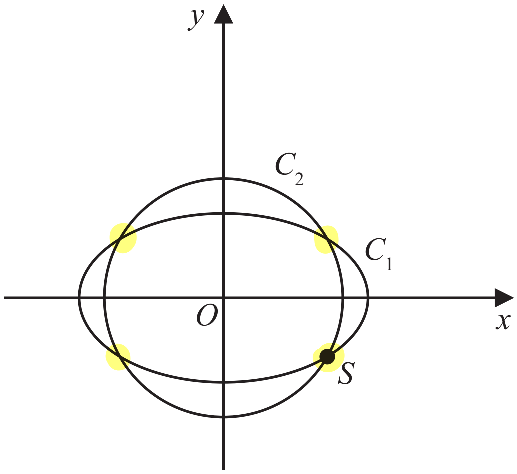


Figure 2

The curve C_1 with parametric equations

$$x = 10\cos t, \quad y = 4\sqrt{2}\sin t, \quad 0 \leqslant t < 2\pi$$

meets the circle C_2 with equation

$$x^2 + y^2 = 66$$

at four distinct points as shown in Figure 2.

Given that one of these points, S, lies in the 4th quadrant, find the Cartesian coordinates of S.

(6)

$C_1 : x = 10\cos t$ and $y = 4\sqrt{2}\sin t$

$C_2 : x^2 + y^2 = 66$

$C_1 \rightarrow C_2$

$(4\sqrt{2})^2 = 32$

$\Rightarrow (10\cos t)^2 + (4\sqrt{2}\sin t)^2 = 66$ (1)

$\Rightarrow 100\cos^2 t + 32\sin^2 t = 66$

$\sin^2 t + \cos^2 t = 1 \Rightarrow \cos^2 t = 1 - \sin^2 t$ (1)

$\Rightarrow 100(1 - \sin^2 t) + 32\sin^2 t = 66$ (1)

$\Rightarrow 100 - 100\sin^2 t + 32\sin^2 t = 66$

$\Rightarrow 68\sin^2 t = 34$

$\Rightarrow 2\sin^2 t = 1$

$\Rightarrow \sin^2 t = 1/2$

$\Rightarrow \sin t = \frac{1}{\sqrt{2}}$ (1) $\Rightarrow t = \sin^{-1}(1/\sqrt{2}) = \frac{\pi}{4}$

$x = 10\cos t$ $\quad y = 4\sqrt{2}\sin t$

$x = 10\cos(\pi/4)$ $\quad y = 4\sqrt{2}\sin(\pi/4)$

$\underline{x = 5\sqrt{2}}$ $\quad y = 4 \Rightarrow y = -4$ since S lies below the y-axis. (1)

$\Rightarrow S = \underline{(5\sqrt{2}, -4)}$ (1)

4. A curve C has parametric equations

$$x = \frac{t^2 + 5}{t^2 + 1} \qquad y = \frac{4t}{t^2 + 1} \qquad t \in \mathbb{R}$$

Show that all points on C satisfy

$$(x - 3)^2 + y^2 = 4$$

(3)

$x = \dfrac{t^2+5}{t^2+1}$ — 1

$y = \dfrac{4t}{t^2+1}$ — 2

$(x-3)^2 + y^2 = 4$ substitute 1 and 2 in here

$\left(\dfrac{t^2+5}{t^2+1} - 3\right)^2 + \left(\dfrac{4t}{t^2+1}\right)^2 = 4$ — 1

$\left(\dfrac{t^2+5-3(t^2+1)}{t^2+1}\right)^2 + \left(\dfrac{4t}{t^2+1}\right)^2 = 4$

$\left(\dfrac{2-2t^2}{t^2+1}\right)^2 + \left(\dfrac{4t}{t^2+1}\right)^2 = 4$

$\dfrac{(2-2t^2)^2 + (4t)^2}{(t^2+1)^2} = 4$

$\dfrac{4 - 8t^2 + 4t^4 + 16t^2}{(t^2+1)^2} = 4$

$\dfrac{4t^4 + 8t^2 + 4}{(t^2+1)^2} = 4$ — 1

$\dfrac{4(t^4+2t^2+1)}{(t^2+1)^2} = 4$

$\dfrac{4\cancel{(t^2+1)^2}}{\cancel{(t^2+1)^2}} = 4$ — 1 $\quad \therefore 4 = 4$

5. The curve C has parametric equations

$$x = \sin 2\theta \qquad y = \operatorname{cosec}^3\theta \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of θ

(3)

(b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$

(3)

a) $y = \operatorname{cosec}^3\theta$

using product rule for brackets

$y = (\operatorname{cosec}\theta)^3$

$\frac{d}{dx}(f(x))^n = f'(x) \times (n-1) \times (f(x))^{n-1}$

$\frac{dy}{d\theta} = 3 \times -\operatorname{cosec}\theta\cot\theta \times (\operatorname{cosec}\theta)^2 = -3\operatorname{cosec}^2\theta\operatorname{cosec}\theta\cot\theta$ ①

$= -3\operatorname{cosec}^3\theta\cot\theta$

$x = \sin 2\theta \qquad \frac{dx}{d\theta} = 2\cos 2\theta$ using differentiation laws for trig

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ ①

$\frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}$ ①

$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

b) $y = 8 \Rightarrow \operatorname{cosec}^3\theta = 8 \Rightarrow \sin^3\theta = \frac{1}{8}$ ①

$\sin\theta = \sqrt[3]{\frac{1}{8}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ remember to use radians!

$\frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\left(\frac{\pi}{6}\right)\cot\left(\frac{\pi}{6}\right)}{2\cos\left(\frac{2\pi}{6}\right)}$ ① (put in calculator)

$\frac{dy}{dx} = -24\sqrt{3}$ ①

6. The curve C has parametric equations

$$x = 2\cos t, \quad y = \sqrt{3}\cos 2t, \quad 0 \leqslant t \leqslant \pi$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(2)

The point P lies on C where $t = \frac{2\pi}{3}$

The line l is the normal to C at P.

(b) Show that an equation for l is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line l intersects the curve C again at the point Q.

(c) Find the exact coordinates of Q.

You must show clearly how you obtained your answers.

(6)

a) $x = 2\cos t$ and $y = \sqrt{3}\cos(2t)$ $\quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ 1

$\frac{dx}{dt} = -2\sin t$ $\quad \frac{dy}{dt} = 2 \times \sqrt{3} \times -\sin(2t)$

$\frac{dy}{dt} = -2\sqrt{3}\sin(2t)$ $\quad \sin(2t) = 2\sin t\cos t$

$= \frac{dy}{dx} = \frac{-2\sqrt{3}\sin(2t)}{-2\sin(t)} = \frac{\sqrt{3}(2\sin t\cos t)}{\sin(t)} = \frac{2\sqrt{3}\cancel{\sin t}\cos t}{\cancel{\sin} t}$

$\Rightarrow \frac{dy}{dx} = \underline{2\sqrt{3}\cos(t)}$ 1

b) $\frac{dy}{dx} = 2\sqrt{3}\cos(t) = 2\sqrt{3}\cos\left(\frac{2\pi}{3}\right) = -\sqrt{3}$ ①

Gradient of the normal $= -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \frac{1}{\sqrt{3}} = m$ ①

$t = \frac{2\pi}{3}$ and $x = 2\cos t$ and $y = \sqrt{3}\cos(2t)$

$\Rightarrow x = 2\cos\left(\frac{2\pi}{3}\right)$ $\qquad y = \sqrt{3}\cos\left(2 \times \frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\Rightarrow x = -1$ ①

$y - \left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{3}}\left(x - (-1)\right) \Rightarrow y = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{2}$ ①

$\Rightarrow y = \frac{1}{\sqrt{3}}x - \frac{\sqrt{3}}{6}$

$\times\sqrt{3}$ $\Rightarrow \sqrt{3}y = x - \frac{1}{2}$

$\times 2$ $\Rightarrow 2\sqrt{3}y = 2x - 1$

$\Rightarrow 2x - 2\sqrt{3}y - 1 = 0$ as required. ①

c) $x = 2\cos t$ $\qquad y = \sqrt{3}\cos(2t)$

Eq of line l: $2x - 2\sqrt{3}y - 1 = 0$

$\Rightarrow 2(2\cos t) - 2\sqrt{3}(\sqrt{3}\cos(2t)) - 1 = 0$ ① $\qquad 6\cos(2t) = 6(2\cos^2 t - 1)$

$\Rightarrow 4\cos t - 6\cos(2t) - 1 = 0$ $\qquad = 12\cos^2 t - 6$

$\Rightarrow 4\cos t - 12\cos^2 t + 6 - 1 = 0$ ①

$\times -1$ $\Rightarrow 12\cos^2 t - 4\cos t - 5 = 0$ ①

Now, let $\theta = \cos t$

$12\theta^2 - 4\theta - 5 = 0 \Rightarrow \theta = \frac{4 \pm \sqrt{(-4)^2 - 4(12)(-5)}}{2 \times 24}$

+ve $\sqrt{\ }$: $\theta = \frac{5}{6}$, -ve $\sqrt{\ }$: $\theta = -\frac{1}{2}$

$\cos t = \frac{5}{6}$ $\qquad \cos t = -\frac{1}{2} \Rightarrow t = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \Rightarrow$ ignore solution. ①

$x = 2\cos t$ and $y = \sqrt{3}\cos(2t)$ $\qquad 2t = \cos^{-1}(5/6) \times 2$

$x = 2 \times \frac{5}{6} = \frac{5}{3}$ $\qquad y = \sqrt{3}\cos\left(\cos^{-1}(5/6) \times 2\right) = \frac{7\sqrt{3}}{18}$ ①

$Q : \left(\frac{5}{3}, \frac{7\sqrt{3}}{18}\right)$ ①